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New extension of some fixed point results in complete metric spaces

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Abstract

We provide some new fixed point results which are inspired by the works of Suzuki and Kannan. The results are proved using the properties of sequentially convergent mappings and A-contractions. Existence and uniqueness of fixed points of self maps satisfying certain conditions are investigated in a complete metric space.

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1 Introduction

In 1968, Kannan [7] proved one of the interesting generalization of the Banach Contraction Principle. The most important fact of the Kannan's result was that the self mapping on a complete metric space need not be continuous. Following the Banach Contraction Principle, Boyd and Wong [3] investigated the fixed point results in nonlinear contraction maps. Subsequently, many authors extended and generalized this fixed point theorem in different points of view.

The study of fixed point results in partially ordered sets was initiated by Ran and Reurings [14]. Their results are hybrid of two classical theorems: Banach fixed point theorem and Knaster-Tarski fixed point theorem. Neito & Rodríguez-López ([11], [12]) extended the main results of Ran and Reurings showing that monotonicity and continuity are not necessary for uniqueness of fixed point.

There have been many exciting developments in the field of existence and uniqueness of fixed points in various directions [1, 5, 6, 8, 9, 10, 13, 16, 17]. Srivastava et al. [18, 19] gave some interesting applications of fixed point theorems in fractional integral equations. Akram et al. [2] gave a characterization for a complete metric space in terms of fixed point property for A-contractions.

Inspired by Kannan [7] and Suzuki [20, 21], in the present paper, we newly extend and generalize some important fixed point results with the help of properties of sequentially convergent mappings and A-contractions.

2 Preliminaries

First we list some important definitions and results which are useful in the present context. Throughout the paper $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ will denote the set of natural numbers, real numbers and non-negative real numbers respectively.

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Definition 2.1. [15] Let $\Phi = \{\varphi | \varphi : \mathbb{R}_+ \to \mathbb{R}_+\}$ be a class of function, which satisfies the following conditions.

(i) $t_1 \leq t_2$ implies $\varphi(t_1) \leq \varphi(t_2)$;

(ii) $(\varphi^n(t))_{n\in N}$ converges to 0 for all $t > 0$;

(iii) $\sum \varphi^n(t)$ converges for all $t > 0$.

If conditions $(i-ii)$ hold, then φ is called a comparison function, and, if the comparison function satisfies (iii), then φ is called a strong comparison function.

Remark 2.2. [15] Any strong comparison function is a comparison function.

Remark 2.3. [15] If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function, then $\varphi(t) < t$, for all $t > 0$, $\varphi(0) = 0$ and φ is right continuous at 0.

Definition 2.4. [2] Let A be the set of all functions $\alpha : \mathbb{R}^3_+ \to \mathbb{R}_+$ satisfying

(i) α is continuous on the set $\mathbb{R}^3_+$ (with respect to the Euclidean metric on $\mathbb{R}^3_+$).

(ii) $a \leq kb$ for some $k \in [0, 1)$ whenever $a \leq \alpha(a, b, b)$ or $a \leq \alpha(b, a, b)$ or $a \leq \alpha(b, b, a)$ for all a, b.

Definition 2.5. [2] A self map T on a metric space X is said to be a A-contraction if it satisfies the condition:

$$d(Tx, Ty) \leq \alpha(d(x, y), d(x, Tx), d(y, Ty))$$

for all $x, y \in X$ and some $\alpha \in A$.

3 Main results

Now we illustrate our main results.

Theorem 3.1. Let X be a complete metric space and $T : X \to X$ be a self map which is continuous such that for every $x \in X$, $\{T^n x\}$ is a Cauchy sequence. Then T has a fixed point.

Further if there is an injective self map $S : X \to X$ with $\alpha \in (0, 1)$ and $\gamma > 0$ such that $2\alpha + \gamma < 1$ and

$$d(STx, STy) \leq \alpha(d(Sx, STx) + d(Sy, STy)) + \gamma d(Sx, Sy),$$

for all $x, y \in X$, then the fixed point is unique and for any $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to the unique fixed point.

Proof. Let $x_0 \in X$ and define the sequence $x_{n+1} = T(x_n)$ for $n = 0, 1, 2, \ldots$. Now, since $\{T^n x_0\}$ is Cauchy and X is complete, there exists $x^{'} \in X$ such that $T^n x_0 \to x^{'}$. Again since $T^n x_0 = x_n$, we have that $x_n \to x^{'}$. By the continuity of T, we have $Tx_n \to Tx^{'}$. Now since the sequences x_n and Tx_n are essentially same, we must have $Tx^{'} = x^{'}$.

Let $x, y \in X$ be fixed points of T, i.e., $Tx = x$ and $Ty = y$. Then

$$\begin{aligned} d(Sx, Sy) &= d(STx, STy) \\ &\leq \alpha(d(Sx, STx) + d(Sy, STy)) + \gamma d(Sx, Sy) \\ &= \alpha(d(Sx, Sx) + d(Sy, Sy)) + \gamma d(Sx, Sy) \\ &= \gamma d(Sx, Sy). \end{aligned}$$

Thus, $d(Sx, Sy) \leq \gamma d(Sx, Sy)$, which is possible only if $d(Sx, Sy) = 0$. Thus we must have $Sx = Sy$. Now S being injective, we have $x = y$.

Finally, since $x_0 \in X$ was arbitrary, this implies that for each $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to the unique fixed point. Q.E.D.

Following example demonstrates Theorem 3.1.

Example 3.2. Let $X = \{0\} \cup \{1, 1/2, 1/3, \ldots\}$ and $d(x, y) = |x - y|$, for all $x, y \in X$. Then (X, d) is a complete metric space. Let the self map $T : X \to X$ be defined as $T(0) = 0$, $T(\frac{1}{n}) = \frac{1}{n+1}$, $n \in \mathbb{N}$. Now, $x_n = \{\frac{1}{n}\}$ is a Cauchy sequence. Also, define the mapping $S : X \to X$ as $S(0) = 0$, $S(\frac{1}{n}) = \frac{1}{[e^n]}$, $n \in \mathbb{N}$. We see that S is continuous, injective and sequentially convergent. If we take $\alpha \in (0, 1)$ and $\gamma > 0$ such that $2\alpha + \gamma < 1$, then for all $x, y \in X$, the conditions of the Theorem 3.1 are satisfied. Thus T has a unique fixed point and the sequence $\{T^n x_0\} = x_n$ converges to the unique fixed point. Moreover, 0 is a unique fixed point of T.

Theorem 3.3. Let X be a complete metric space and $T : X \to X$ be a self map which is continuous such that for every $x \in X$, $\{T^n x\}$ is a Cauchy sequence. Then T has a fixed point.

Further if there is an injective self map $S : X \to X$ with $\alpha \in (0, 1)$ and $\gamma > 0$ such that $2\alpha + \gamma < 1$ and

$$d(STx, STy) \leq \alpha(d(Sx, STy) + d(Sy, STx)) + \gamma d(Sx, Sy),$$

for all $x, y \in X$, then the fixed point is unique and for any $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to the unique fixed point.

Proof. The first part can be proved using similar techniques as in Theorem 3.1.

To prove the uniqueness, let $x, y \in X$ be fixed points of T, i.e., $Tx = x$ and $Ty = y$. Then

$$\begin{aligned} d(Sx, Sy) &= d(STx, STy) \\ &\leq \alpha(d(Sx, STy) + d(Sy, STx)) + \gamma d(Sx, Sy) \\ &= \alpha(d(Sx, Sy) + d(Sy, Sx)) + \gamma d(Sx, Sy) \\ &= \alpha(2d(Sx, Sy)) + \gamma d(Sx, Sy) \\ &= (2\alpha + \gamma) d(Sx, Sy) \\ \Rightarrow d(Sx, Sy) &\leq (2\alpha + \gamma) d(Sx, Sy). \end{aligned}$$

Since $(2\alpha + \gamma) \leq 1$, we have $d(Sx, Sy) = 0$. Thus $Sx = Sy$. Now S being injective, we get $x = y$.

Finally, since $x_0 \in X$ was arbitrary, this implies that for each $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to the unique fixed point.

Q.E.D.

Next, we establish some fixed point results using the concept of sequentially convergent maps as given below.

Definition 3.4. [4] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be sequentially convergent if, for each sequence $\{y_n\}$, the following holds true:
if $\{Ty_n\}$ is convergent, then $\{y_n\}$ is also convergent.

Theorem 3.5. Let (X, d) be a complete metric space and $T : X \to X$, $S : X \to X$ be self maps on X such that S is continuous, sequentially convergent and injective satisfying

$$d(STx, STy) \leq \alpha(d(Sx, Sy),$$

for all $x, y \in X$, where $\alpha \in (0, 1)$. Then T has a unique fixed point.
Further, for any $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to the fixed point.

Proof. Let $x_0 \in X$. Define the sequence $x_{n+1} = T(x_n)$ for $n = 0, 1, 2, \ldots$. Then we have $T^n x_0 = x_n$ for all $n \in \mathbb{N}$. From the given condition

$$d(STx, STy) \leq \alpha d(Sx, Sy),$$
$$\text{we have } d(STx_n, STx_{n-1}) \leq \alpha d(Sx_n, Sx_{n-1}).$$

This implies,

$$d(Sx_{n+1}, Sx_n) \leq \alpha d(Sx_n, Sx_{n-1}). \tag{3.1}$$

Now, from 3.1, we have for all $m, n \in \mathbb{N}$ with $r > m$,

$$d(Sx_n, Sx_m) < \alpha^m d(Sx_1, Sx_0).$$

If $m, n \to \infty$, then $d(Sx_n, Sx_m) \to 0$ and consequently $\{Sx_n\}$ is a Cauchy sequence. X being complete, $\{Sx_n\}$ must be convergent. Again S being sequentially convergent, x_n must be convergent to some $x \in X$ as well. Since S is continuous, $Sx_n \to Sx$.

Now,

$$\begin{aligned} d(STx, Sx) &\leq d(STx, Sx_n) + d(Sx_n, Sx_{n+1}) + d(Sx_{n+1}, Sx) \\ &= d(STx_n, ST^n x_0) + d(Sx_n, Sx_{n+1}) + d(Sx_{n+1}, Sx) \\ &\leq \alpha(d(Sx, Sx_{n-1}) + \alpha^n d(Sx_0, Sx_1) + d(Sx_{n+1}, Sx). \end{aligned}$$

Taking limit as $n \to \infty$, we have

$$d(STx, Sx) \leq \alpha d(Sx, Sx_{n-1}). \tag{3.2}$$

Since $Sx_n \to Sx$, the inequality 3.2 implies that $d(STx, Sx) = 0$. This implies $STx = Sx$ and hence $Tx = x$ (since S is injective). Therefore, x is a fixed point of T.

To prove the uniqueness let $x, y \in X$ be fixed points of T, i.e., $Tx = x$ and $Ty = y$.

Now,

$$d(Sx, Sy) = d(STx, STy) \leq \alpha d(Sx, Sy).$$

Since $\alpha \in (0, 1)$, the above inequality is possible only if $d(Sx, Sy) = 0$, i.e., $Sx = Sy$. Now S being injective, we have $x = y$.

Finally, since $x_0 \in X$ was arbitrary, for every such x_0, the sequence $\{T^n x_0\}$ must converge to the unique fixed point. Q.E.D.

Our next example verifies all conditions of Theorem 3.5.

Example 3.6. Let $X = \{0\} \cup \{1/2, 1/3, \ldots\}$ and $d(x, y) = |x - y|$, for all $x, y \in X$. Then (X, d) is a complete metric space. Consider the self map $T : X \to X$ defined by $Tx = x^3$, for all $x \in X$. Also, let the mapping $S : X \to X$ be defined as $S(0) = 0$, $S(\frac{1}{n}) = \frac{1}{n^2}$, $n \in \mathbb{N} \setminus \{1\}$. Then S is continuous, injective and sequentially convergent. If we consider $\alpha \in (0, 1)$, so that for all $x, y \in X$, the conditions of Theorem 3.5 are satisfied, then T has a unique fixed point and the sequence $\{T^n x_0\}$ converges to the unique fixed point. Here, 0 is the unique fixed point of T.

Theorem 3.7. Let F_1 denote all the continuous functions $\psi : [0, \infty)^2 \to [0, \infty)$ satisfying $\psi(x, y) = 0$ if and only if $x = y = 0$. Also let (X, d) be a complete metric space and $T : X \to X$, $S : X \to X$ be self maps on X such that S is continuous, sequentially convergent and injective satisfying

$$d(STx, STy) \leq \frac{1}{2}[(d(Sx, STx) + d(Sy, STy)] - \psi(d(Sx, STx), d(Sy, STy)),$$

for all $x, y \in X$ and $\psi \in F_1$. Then T has a unique fixed point.

Proof. Let $ST = f$. It is clear that $fX \subseteq SX$. Now, the given contractive condition becomes

$$d\left(fx, fy\right) \leq \frac{1}{2}\left[d\left(Sx, fx\right) + d\left(Sy, fy\right)\right] - \psi\left(d\left(Sx, fx\right), d\left(Sy, fy\right)\right). \tag{3.3}$$

The condition 3.3 for any $x_0 \in X$ (because $fX \subseteq SX$), generates tho so-called Jungck sequence $y_n = fx_n = Sx_{n+1}$. Putting $x = x_n, y = x_{n+1}$ in 3.3 we get

$$d\left(fx_n, fx_{n+1}\right) \leq \frac{1}{2}\left[d\left(Sx_n, fx_n\right) + d\left(Sx_{n+1}, fx_{n+1}\right)\right] - \psi\left(d\left(Sx_n, fx_n\right), d\left(Sx_{n+1}, fx_{n+1}\right)\right),$$

i.e.,

$$d\left(y_n, y_{n+1}\right) \leq \frac{1}{2}\left[d\left(y_{n-1}, y_n\right) + d\left(y_n, y_{n+1}\right)\right] - \psi\left(d\left(y_{n-1}, y_n\right), d\left(y_n, y_{n+1}\right)\right). \tag{3.4}$$

Hence, we obtain that

$$d\left(y_n, y_{n+1}\right) \leq d\left(y_{n-1}, y_n\right), \tag{3.5}$$

for all $n = 1, 2, 3, ...$ Further 3.5 implies that $d\left(y_n, y_{n+1}\right) \to d^* \geq 0$ as $n \to \infty$. If $d^* > 0$ then from 3.4, it follows that $\psi\left(d^*, d^*\right) = 0$, i.e., $d^* = 0$, which is a contradiction. Now, if $\{y_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and sequences $\{n_k\}$ and $\{m_k\}$ of positive integers such that

$$n_k > m_k > k,\ d\left(y_{m_k}, y_{n_k - 1}\right) < \varepsilon,\ d\left(y_{m_k}, y_{n_k}\right) \geq \varepsilon,$$

for all positive integers k. Then,

$$\varepsilon \leq\ d\left(y_{m_k}, y_{n_k}\right) \leq d\left(y_{m_k}, y_{n_k - 1}\right) + d\left(y_{n_k - 1}, y_{n_k}\right). \tag{3.6}$$

From 3.6, it follows that $d\left(y_{m_k}, y_{n_k}\right) \to \varepsilon^+$ as $k \to \infty$. If we take $x = x_{m_k}, y = x_{n_k}$ in 3.3, we get the next relation:

$$d\left(y_{m_k}, y_{n_k}\right) \leq \frac{1}{2}\left[d\left(y_{m_k - 1}, y_{m_k}\right) + d\left(y_{n_k - 1}, y_{n_k}\right)\right] - \psi\left(d\left(y_{m_k - 1}, y_{m_k}\right), d\left(y_{n_k - 1}, y_{n_k}\right)\right). \tag{3.7}$$

Passing to the limit in 3.7 when $k \to \infty$, we obtain

$$\varepsilon \leq \frac{1}{2}\left[0+0\right] - \psi\left(0,0\right) = 0,$$

which is, a contradiction. Hence the sequence $\{Sx_n\}$ is a Cauchy sequence.

Since X is complete, $\{Sx_n\}$ is convergent. Also S is sequentially convergent and since $\{Sx_n\}$ is convergent, it implies that $\{x_n\}$ is convergent. i.e., there exists $x \in X$, so that $lim_{n\to\infty}x_n = x$. By the continuity on S, we have $lim_{n\to\infty}Sx_n = Sx$.

Now,

$$\begin{aligned} d(STx, Sx) &\leq d(STx, STx_{n+1}) + d(STx_{n+1}, Sx) \\ &\leq \frac{1}{2}[(d(Sx, STx) + d(Sx_{n+1}, STx_{n+1})] - \psi(d(Sx, STx), d(Sx_{n+1}, STx_{n+1})) \\ &= \frac{1}{2}[(d(Sx, STx) + d(Sx_{n+1}, Sx_{n+2})] - \psi(d(Sx, STx), d(Sx_{n+1}, Sx_{n+2})). \end{aligned}$$

Taking limit as $n \to \infty$ and using $Sx_n \to Sx$, continuity of ψ, we have

$$\begin{aligned} d(STx, Sx) &\leq \frac{1}{2}d(Sx, STx) - \psi(d(Sx, STx), 0) \\ &= \frac{1}{2}d(Sx, STx), \end{aligned}$$

which is a contradiction unless $d(Sx, STx) = 0$, i.e., $STx = Sx$. Thus $Tx = x$ (since S is injective). Therefore, x is a fixed point of T.

To prove the uniqueness let $Tx = x$ and $Ty = y$. Now,

$$\begin{aligned} d(Sx, Sy) &= d(STx, STy) \\ &\leq \frac{1}{2}[(d(Sx, STx) + d(Sy, STy)] - \psi(d(Sx, STx), d(Sy, STy)) \\ &= \frac{1}{2}[(d(Sx, Sx) + d(Sy, Sy)] - \psi(d(Sx, Sx), d(Sy, Sy)) \\ &= 0 \\ \Rightarrow d(Sx, Sy) &\leq 0. \end{aligned}$$

Hence $d(Sx, Sy) = 0$. Thus $Sx = Sy$. This implies $x = y$ (since S is injective). Therefore, T has a unique fixed point. Q.E.D.

Definition 3.8. Let A_φ be the set of all functions $\alpha : \mathbb{R}_+^3 \to \mathbb{R}_+$ satisfying

(i) α is continuous on the set $\mathbb{R}_+^3$ (with respect to the Euclidean metric on $\mathbb{R}_+^3$).

(ii) for all $u, v \in \mathbb{R}_+$, $u \leq \alpha(u, v, v)$ or $u \leq \alpha(v, u, v)$ or $u \leq \alpha(v, v, u)$, then $u \leq \varphi(v)$, where φ is a strong comparison function.

When $\varphi(t) = kt$ as $k \in (0, 1)$ for all $t > 0$, then $\alpha \in A$.

Theorem 3.9. Let (X, d) be a complete metric space and $T : X \to X$, $S : X \to X$ be self maps on X such that S is continuous, sequentially convergent and injective. If there exists some $\alpha \in A_\varphi$ such that

$$d(STx, STy) \leq \alpha(d(Sx, Sy), d(Sx, STx), d(Sy, STy))$$

for all $x, y \in X$. Then T has a unique fixed point.

Further for any $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to the unique fixed point.

Proof. Let $x_0 \in X$ and define the sequence $\{x_n\}$ as $x_{n+1} = T(x_n)$ for $n = 0, 1, 2, \ldots$. Then we have $T^n x_0 = x_n$ for all $n \in \mathbb{N}$. Now,

$$\begin{aligned} d(Sx_{n+1}, Sx_n) &= d(STx_n, STx_{n-1}) \\ &\leq \alpha(d(Sx_n, Sx_{n-1}), d(Sx_n, STx_n), d(Sx_{n-1}, STx_{n-1})) \\ &= \alpha(d(Sx_{n-1}, Sx_n), d(Sx_n, Sx_{n+1}), d(Sx_{n-1}, Sx_n)) \end{aligned}$$

$$\begin{aligned} \Rightarrow d(Sx_{n+1}, Sx_n) &\leq \alpha(d(Sx_{n-1}, Sx_n), d(Sx_n, Sx_{n+1}), d(Sx_{n-1}, Sx_n)) \\ &\leq \varphi(d(Sx_{n-1}, Sx_n)). \end{aligned} \quad (3.8)$$

Continuing this way, we get

$$\begin{aligned} d(Sx_{n+1}, Sx_n) &\leq \varphi(d(Sx_{n-1}, Sx_n)) \\ &\leq \varphi(\varphi(d(Sx_{n-2}, Sx_{n-1})) \\ &= \varphi^2(d(Sx_{n-2}, Sx_{n-1})) \\ &. \\ &. \\ &. \\ &\leq \varphi^n(d(Sx_0, Sx_1)). \end{aligned}$$

Since, $d(Sx_0, Sx_1) \geq 0$, from the definition 2.1 (ii), we have $lim_{n\to\infty}\varphi^n(d(Sx_0, Sx_1)) = 0$. Now for a given $\varepsilon > 0$, there is a positive integer n_0 such that for all $n \geq n_0$,

$$\varphi^n(d(Sx_0, Sx_1) < \varepsilon - \varphi(\varepsilon).$$

Hence

$$d(Sx_{n+1}, Sx_n) < \varepsilon - \varphi(\varepsilon). \quad (3.9)$$

Now, for any $m, n \in \mathbb{N}$ with $m > n \geq n_0$, we claim that

$$d(Sx_n, Sx_m) < \varepsilon. \quad (3.10)$$

We prove the claim by induction on m. The inequality holds for $m = n+1$ by using condition 3.9. Assume that inequality 3.10 holds for $m = k$, i.e. $d(Sx_n, Sx_k) < \varepsilon$. Now if $m = k + 1$, we have

$$\begin{aligned} d(Sx_n, Sx_{k+1}) &\leq d(Sx_n, Sx_{n+1}) + d(Sx_{n+1}, Sx_{k+1}) \\ &< \varepsilon - \varphi(\varepsilon) + \varphi(d(Sx_n, Sx_k))(\text{since } k = m > n \text{ and using condition } 3.8) \end{aligned}$$

$$
\begin{aligned}
&< \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon) \\
&= \varepsilon.
\end{aligned}
$$

By induction on m, we conclude that the inequality 3.10 holds for $m > n \geq n_0$.

Thus $\{Sx_n\}$ is a Cauchy sequence. Since X is complete, so $\{Sx_n\}$ is convergent. Also, S is sequentially convergent, and since $\{Sx_n\}$ is convergent, it implies that $\{x_n\}$ is convergent, i.e., there exists $x \in X$ so that $lim_{n\to\infty} x_n = x$. Applying continuity on S, we have $lim_{n\to\infty} Sx_n = Sx$.

Now,

$$
\begin{aligned}
d(STx, Sx) &\leq d(STx, STx_{n+1}) + d(STx_{n+1}, Sx) \\
&\leq \alpha(d(Sx, Sx_{n+1}), d(Sx, STx), d(Sx_{n+1}, STx_{n+1})) + d(STx_{n+1}, Sx) \\
&= \alpha(d(Sx, Sx_{n+1}), d(Sx, STx), d(Sx_{n+1}, Sx_{n+2})) + d(Sx_{n+2}, Sx).
\end{aligned}
$$

Taking limit as $n \to \infty$, we have

$$
\begin{aligned}
d(STx, Sx) &\leq \alpha(d(Sx, Sx), d(Sx, STx), d(Sx, Sx)) + d(Sx, Sx) \\
&= \alpha(0, d(Sx, STx), 0) \\
\Rightarrow d(STx, Sx) &\leq \varphi(0) = 0 \\
\Rightarrow STx &= Sx \\
\Rightarrow Tx &= x \text{ (Since S is injective) .}
\end{aligned}
$$

Therefore, x is a fixed point of T. Finally, we prove that x is a unique fixed point of T. Let $Tx = x$ and $Ty = y$. Now,

$$
\begin{aligned}
d(Sx, Sy) &= d(STx, STy) \\
&\leq \alpha(d(Sx, Sy), d(Sx, STx), d(Sy, STy)) \\
&\leq \alpha(d(Sx, Sy), d(Sx, Sx), d(Sy, Sy)) \\
&\leq \alpha(d(Sx, Sy), 0, 0) \\
\Rightarrow d(Sx, Sy) &\leq \alpha(d(Sx, Sy), 0, 0) \\
\Rightarrow d(Sx, Sy) &\leq \varphi(0) = 0 \\
\Rightarrow Sx &= Sx \\
\Rightarrow x &= y \text{ (Since S is injective) .}
\end{aligned}
$$

Finally, since $x_0 \in X$ was arbitrary, for every such x_0 the sequence $\{T^n x_0\}$ must converge to the unique fixed point. Q.E.D.

Corollary 3.10. Let (X, d) be a complete metric space and $T : X \to X$, $S : X \to X$ be self maps on X such that S is continuous, sequentially convergent and injective. If there exists some $\alpha \in A$ such that

$$d(STx, STy) \leq \alpha(d(Sx, Sy), d(Sx, STx), d(Sy, STy))$$

for all $x, y \in X$. Then T has a unique fixed point.

Conclusion. In this paper, by using the properties of sequentially convergent mappings and A-contractions, we have produced new extensions of some of important fixed point results similar to those of Kannan and Suzuki. The results obtained in this paper are mainly concerned with the existence and uniqueness of fixed point. Our results, eventually, would extend and generalize the existing results in similar context. In our results, we have assumed the continuity of the self-map T. It would be an interesting topic for future study if the continuity of the map can be relaxed. Some well known theorems about common fixed point may also be generalized in the present context in future.

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